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UMUMIY O‘RTA TA’LIM MAKTABLARIDA PARAMETR QATNASHGAN TRIGONOMETRIK TENGLAMALARNI O‘RGANISH

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Maqola parametr qatnashgan trigonometrik tenglamalarni o‘rganishga bag‘ishlangan. Unda o‘quvchilarni $\sin x = a, |a| \le 1$, $\cos x = a, |a| \le 1$, $tgx = a$ *va* $ctgx = a$ *ko‘rinishidagi oddiy trigonometrik tenglamalarni turli a qiymatlarida yecha olishga o‘rgatish masalasi ko‘rib chiqilgan. O‘quvchilar parametrli tenglamalarni yechishda trigonometrik formulalar va ayniyatlardan foydalanib, teng kuchlilik shartlarini bajarishi hamda yechimlarni tekshirishi haqida so‘z boradi.*

Kalit so‘zlar: *tenglama, parametr, parametr qatnashgan trigonometrik tenglama, tenglamaning yechimi, trigonometrik ayniyat, tengsizlik, parametrning mumkin bo‘lgan qiymatlari.*

Статья посвящена изучению тригонометрических уравнений с параметром. Подчёркивается, что ученики должны уметь решать простейшие тригонометрические уравнения, такие как $\sin x = a, |a| \le 1$, $\cos x = a, |a| \le 1$, $tgx = a$ *и* $ctgx = a$, *при различных значениях параметра a. Также в статье отмечается важность использования тригонометрических формул и тождеств для решения уравнений с параметром, при этом следует проверять выполнение условий равносильности и правильность полученных решений*.

Ключевые слова: *уравнение, параметр, тригонометрические уравнение с параметрами, решение уравнений, тригонометрическое тождество, неравенство, допустимое значение параметра.*

Ma’lumki, maktab matematika kursidagi bir qator mavzularni o‘zlashtirishda o‘quvchilar qiyinchiliklarga duch keladilar. Ularga misol sifatida modulli tenglama va tengsizliklar hamda parametr qatnashgan tenglama va tengsizliklarni keltirish mumkin. Bunday mavzular maktab darsliklarida alohida berilmagan. Lekin har qanday tenglama va tengsizliklarni o‘rganish modul va parametr qatnashgan tenglama hamda tengsizliklarni o‘rganish bilan yakunlanadi.

Tenglamalar ichida parametr qatnashgan trigonometrik tenglamalar ham o‘quvchilar qiyin o‘zlashtiradigan mavzular sarasiga kirishini yuqorida ta’kidladik. O‘quvchilar bunday tenglamalarni yechishni puxta o‘zlashtirishlari uchun eng avvalo parametr qatnashmagan trigonometrik tenglamalarni yechishni puxta o‘zlashtirishlari kerak bo‘ladi. Buning uchun esa ular $\sin x = a, |a| \le 1$, $\cos x = a, |a| \le 1$, $tgx = a$ va $ctgx = a$ ko‘rinishidagi eng sodda trigonometrik tenglamalarni a ning ko‘p uchrab turadigan turli tayin qiymatlaridagi yechimlarini yoza bilishlari kerak. Ularni quyida berilgan jadvaldan hosil qilish mumkin.

	$\sin x = a$	$\cos x = a$
$\|a\|<1$	$x = (-1)^k arcsin a + k\pi,\ k \in Z$	$x = \pm arccos a + 2\pi k,\ k \in Z$
$a > 1$	$x = \emptyset$	$x = \emptyset$
$a < -1$	$x = \emptyset$	$x = \emptyset$
$a = 1$	$x = \frac{\pi}{2} + 2\pi k,\ k \in Z$	$x = 2\pi k,\ k \in Z$
$a = -1$	$x = -\frac{\pi}{2} + 2\pi k,\ k \in Z$	$x = \pi + 2\pi k,\ k \in Z$
$a \in R$	$tgx = a\ x = arctga + \pi k,\ k \in Z$	
$a \in R$	$ctgx = a\ x = arcctga + \pi k,\ k \in Z$	

Parametr qatnashgan eng sodda trigonometrik tenglamalar ko‘pincha $\sin[f(x)] = a$, $|a| \le 1$; $\cos[f(x)] = a$, $|a| \le 1$; $tg\,[f(x)] = a$; $ctg\,[f(x)] = a$ ko‘rinishida ham yoziladi. Bu yerda $f(x)$ x ga bog‘liq ifoda. Ularning yechimlari umumiy holda quyidagicha yoziladi:

1. $\sin[f(x)] = a, |a| \le 1$ tenglama $f(x) = (-1)^k arcsina + k\pi$ ga teng kuchli.
2. $\cos[f(x)] = a, |a| \le 1$ tenglama $f(x) = \pm arccosa + 2\pi k$ ga teng kuchli.
3. $tg[f(x)] = a$ tenglama $f(x) = arctga + \pi k$ ga teng kuchli.
4. $ctg[f(x)] = a$ tenglama $f(x) = arcctga + \pi k$ ga teng kuchli.

Bu yechimlardagi z butun sonlar to‘plamidan qiymatlar qabul qiladi.

Parametr qatnashgan trigonometrik tenglamalarni yechishda trigonometrik formulalar va asosiy trigonometrik ayniyatlardan foydalangan holda ba’zi bir shakl almashtirishlar bajariladi. Agar bunday almashtirishlar tenglamalarning teng kuchliligiga doir teoremalarga asoslangan bo‘lsa, u holda hosil bo‘lgan tenglamaning yechimi berilgan tenglamaning yechimi bo‘ladi. Aks holda yechimlar tekshirilishi kerak. Parametr qatnashgan trigonometrik tenglamalarni yechishda parametrning mumkin bo‘lgan qiymatlari tushunchasi muhimdir. Masalan, $\sin\left(2x + \frac{\pi}{3}\right) = a$, $\cos\left(5x + \frac{\pi}{3}\right) = a$ tenglamalarda a parametr $|a| \le 1$ tengsizlikni qanoatlantirishi kerak bo‘lsa, $tg\left(\frac{1}{3}x + \frac{\pi}{6}\right) = a$ va $ctg\left(2x + \frac{\pi}{4}\right) = a$ tenglamalarda a parametr ixtiyoriy qiymatlarni qabul qilishi mumkin.

Quyida parametr qatnashgan trigonometrik tenglamalarga misollar keltiramiz.

1. $\sin(2x+3) = b + 1$ tenglama yechilsin.

Yechish: Berilgan tenglama $\sin[f(x)] = a$ ko‘rinishidagi tenglamadir.

Bu yerda $f(x) = 2x+3$ ga $a = b+1$ bo‘lishi ravshan. Berilgan tenglama $-1 \le b + 1 \le 1$ da, ya’ni $-2 \le b \le 0$ shartda $2x + 3 = (-1)^k arcsina + k\pi$, $k \in Z$, ya’ni $x = -1{,}5 + 0{,}5 \cdot (-1)^k arcsin(b + 1) + 0{,}5\pi$; yechimga ega.

Javob: Agar $-2 \le b \le 0$ bo‘lsa, $x = -1{,}5 + 0{,}5 \cdot (-1)^k arcsin(b + 1) + 0{,}5\pi$; agar $b < -2$ va $b > 0$ bo‘lsa, $x \in 0$.

2. $tg^2 2x - (2a + 1)tg2x + a(a + 1) = 0$ tenglama yechilsin.

Yechish: Berilgan tenglama $tg2x$ ga nisbatan kvadrat tenglamadir. Uni yechib, $tg2x = a + 1$ va $tg2x = a$ lardan iborat ikkita $tg[f(x)] = a$ ko‘rinishidagi eng sodda tenglamalarga ega bo‘lamiz.

Ulardan $x_1 = \frac{1}{2} arctg(a + 1) + \frac{1}{2}\pi k$ va $x_2 = \frac{1}{2} arctga + \frac{1}{2}\pi k$, $k \in Z$ yechimlarni topamiz.

Javob: $\frac{1}{2} arctg(a + 1) + \frac{1}{2}\pi k$, $k \in Z$; $\frac{1}{2} arctga + \frac{1}{2}\pi k$, $k \in Z$.

3. $\cos(x + a) = \frac{\cos a}{\cos x}$ tenglama yechilsin.

Yechish: Dastlab berilgan tenglamaning har ikkala tomonini $\cos x$ ga ko‘paytiramiz, so‘ngra hosil bo‘lgan tenglamaning chap tomonidagi ko‘paytmani yig‘indiga almashtiramiz.

$$\cos(x + a) = \frac{\cos a}{\cos x},\ \cos(x + a) \cdot \cos x = \cos a,$$
$$\cos(x + a + x) + \cos(x + a - x) = 2\cos a,$$
$$\cos(2x + a) + \cos a = 2\cos a,\ \cos(2x + a) = \cos a.$$

Oxirgi tenglamadan $x = k\pi$ va $x = -a + k\pi$ larni topamiz.

Endi hosil bo‘lgan yechimlardan $\cos x = 0$ tenglamaning yechimlarini chiqarib tashlaymiz. U yechimlar $x = \frac{\pi}{2} + n\pi$ dan iborat.

$x = k\pi$ va $x = \frac{\pi}{2} + n\pi$ yechimlar bir-biri bilan kesishmaydi.

Navbatda $-a + k\pi = \frac{\pi}{2} + n\pi$ dan $a = \frac{\pi}{2}(-1 - 2n + 2k)$ ni topamiz. Bundan $x = -a + k\pi$ berilgan tenglamaning yechimi bo‘lishi uchun $a \ne \frac{\pi}{2}(2n - 2k - 1)$ bo‘lishi kerakligini topamiz.

Javob: Agar $a = \frac{\pi}{2}(2n - 2k - 1)$ bo‘lsa, u holda $x = k\pi$; agar $a \ne \frac{\pi}{2}(2n - 2k - 1)$ bo‘lsa, u holda $x = k\pi$; $x = -a + k\pi$.

4. $\sin^4 x + \cos^4 x = a$ tenglama yechilsin.

Yechish: Darajani pasaytirish formulalaridan foydalanamiz.

$$\sin^2 x = \frac{1-\cos 2x}{2},\ \cos^2 x = \frac{1+\cos 2x}{2},\ \left(\frac{1-\cos 2x}{2}\right)^4 + \left(\frac{1+\cos 2x}{2}\right)^4 = a,$$

$$2\cos^2 2x = 4a - 2,\ \cos^2 2x = 2a - 1,\ \frac{1+\cos 4x}{2} = 2a - 1,$$

$$1 + \cos 4x = 4a - 2,\ \cos 4x = 4a - 3.$$

Hosil bo'lgan tenglama parametr qatnashgan eng sodda trigonometrik tenglamadir. Parametrning mumkin bo'lgan qiymatlari $|4a-3| \le 1;\ -1 \le 4a-3 \le 1,2 \le 4a \le 4,\ \frac{1}{2} \le a \le 2$ dan iborat.

Demak, biz $a < \frac{1}{2},\ a = \frac{1}{2},\ \frac{1}{2} < a < 1,\ a = 1,\ a > 1$ hollarini qarashimiz mumkin.

1) agar $a < \frac{1}{2}$ bo'lsa, u holda $2a - 1 < 0$ bo'ladi va $\cos^2 2x = 2a - 1$ tenglama yechimga ega bo'lmaydi;

2) agar, $a = \frac{1}{2}$ bo'lsa, u holda $\cos^2 2x = 0$ bo'lib, undan $x = \frac{\pi}{4} + \frac{k\pi}{4}$ ni topamiz;

3) agar $\frac{1}{2} < a < 1$ bo'lsa, u holda $0 < 2a - 1 < 1$. Bu holda $\cos^2 2x = 2a - 1$ dan $\cos 4x = 4a - 3$ kelib chiqadi. Bundan esa $4x = \pm\, arccos(4a-3) + 2\pi k$ bo'lib, bundan $x = \pm\frac{1}{4}\, arccos(4a-3) + \frac{\pi k}{2}$;

4) agar $a = 1$ bo'lsa, u holda $\cos^2 2x = 1$ bo'ladi va bundan $x = \frac{\pi k}{2}$;

5) agar $a > 1$ bo'lsa, $2a - 1 > 1$ bo'ladi va bu holda yechim yo'q.

Javob: Agar $a < \frac{1}{2},\ a > 1$ bo'lsa yechim yo'q; agar $\frac{1}{2} < a < 1$ bo'lsa, $x = \pm\frac{1}{4}\, arccos(4a-3) + \frac{\pi k}{2}$.

5. $a \sin^2 x + \cos x = 0$ tenglama yechilsin.

Yechish: $\sin^2 x = 1 - \cos^2 x$ ekanligidan foydalanamiz.

$$a \sin^2 x + \cos x = 0,$$
$$a(1 - \cos^2 x) + \cos x = 0,$$
$$-a \cos^2 x + \cos x + a = 0,$$
$$a \cos^2 x - cos\, x - a = 0.$$

$\cos x = y$ deylik, u holda $ay^2 - y - a = 0$ tenglama hosil bo'ladi. Uni yechamiz:

$$y_{1,2} = \frac{1 \pm \sqrt{1+4a^2}}{2a};\ y_1 = \frac{1-\sqrt{1+4a^2}}{2a};\ y_2 = \frac{1+\sqrt{1+4a^2}}{2a}$$

Demak, biz $\cos x = \frac{1-\sqrt{1+4a^2}}{2a}$ va $\cos x = \frac{1+\sqrt{1+4a^2}}{2a}$ lardan iborat 2ta tenglamaga ega bo'ldik. Quyida bu tenglamalarni yechamiz.

$$\cos x = \frac{1-\sqrt{1+4a^2}}{2a},\ -1 \le \frac{1-\sqrt{1+4a^2}}{2a} \le 1,$$

$$-2a \le 1 - \sqrt{1+4a^2} \le 2a,\ -2a - 1 \le -\sqrt{1+4a^2} \le 2a - 1,$$

$$-2a + 1 \le \sqrt{1+4a^2} \le 2a + 1,$$

$$4a^2 - 2a + 1 \le 1 + 4a^2 \le 4a^2 + 4a + 1,$$

$$-2a + 1 \le 1 \le 4a + 1,$$

$$\begin{cases} -2a + 1 \le 1 \\ 4a + 1 \ge 1 \end{cases};\ \begin{cases} 2a \ge 0 \\ 4a > 0 \end{cases};\ a > 0$$

Agar $a \ne 0$ bo'lsa, $x = \pm\, arc\cos\frac{1-\sqrt{1+4a^2}}{2a} + 2k\pi$

Agar $a = 0$ bo'lsa, berilgan tenglama $\cos x = 0$ ko'rinishga keladi. Bundan esa $x = \frac{\pi}{2} + k\pi,\ k \in z$.

$\cos x = \frac{1+\sqrt{1+4a^2}}{2a}$ tenglama yechimga ega emas.

Javob: Agar $a \neq 0$ bo‘lsa, $x = \pm arc\ cos\ \frac{1-\sqrt{1+4a^2}}{2a} + 2k\pi$,

agar $a = 0$ bo‘lsa, $x = \frac{\pi}{2} + k\pi,\ k \in z$.

6. $\sin 2x + 3\cos 2x = a$ tenglama yechilsin.

Yechish: Berilgan tenglamaning har ikkala tomonini $\frac{1}{\sqrt{10}}$ ga ko‘paytiramiz. $\frac{1}{\sqrt{10}}\sin 2x + \frac{3}{\sqrt{10}}\cos 2x = \frac{a}{\sqrt{10}}$.

Agar $\sin\varphi = \frac{1}{\sqrt{10}}$ deb olsak, u holda $\cos\varphi = \frac{3}{\sqrt{10}}$ bo‘ladi va berilgan tenglama quyidagi ko‘rinishga keladi:

$\sin 2x \cdot \sin\varphi + \cos 2x \cdot \cos\varphi = \frac{a}{\sqrt{10}},\ \cos(2x-\varphi) = \frac{a}{\sqrt{10}}$.

Bu tenglama $\left|\frac{a}{\sqrt{10}}\right| \leq 1$ bo‘lganda yechimga ega. Bundan $-1 \leq \frac{a}{\sqrt{10}} \leq 1$ *yoki* $-\sqrt{10} \leq a \leq \sqrt{10}$ kelib chiqadi. Bu shartda oxirgi tenglamadan quyidagiga ega bo‘lamiz:

$$2x - \varphi = \pm arc\cos\frac{a}{\sqrt{10}} + 2k\pi,\ 2x = \varphi \pm arc\cos\frac{a}{\sqrt{10}} + 2k\pi,$$

$x = \frac{\varphi}{2} \pm \frac{1}{2} arc\cos\frac{a}{\sqrt{10}} + k\pi$. Endi φ ni aniqlaymiz. $\sin\varphi = \frac{1}{\sqrt{10}}$ va $\cos\varphi = \frac{3}{\sqrt{10}}$ bo‘lganligi uchun $tg\,\varphi = \frac{1}{3}$ bo‘lib, bundan $\varphi = ctg\frac{1}{3}$. Demak berilgan tenglamaning umumiy yechimi quyidagicha bo‘ladi:

$$x = \pm\frac{1}{2} arc\cos\frac{a}{\sqrt{10}} + \frac{1}{2} arc\,\mathrm{tg}\frac{1}{3} + k\pi,\ k \in z \text{ agar } a \in (-\sqrt{10};\ \sqrt{10}].$$

7. $2\cos^2 6x - 9\sin^2 6x + 4\sin 6x\cos 6x = a + 5$ tenglama yechilsin.

Yechish: $\cos^2 6x = \frac{1+\cos 12x}{2},\ \sin^2 6x = \frac{1-\cos 12x}{2}$,

$$\sin 6x\cos 6x = \frac{1}{2}\sin 12x$$

bo‘lganligi uchun berilgan tenglamadan quyidagi tenglama kelib chiqadi:

$2 \cdot \frac{1+\cos 12x}{2} - 9 \cdot \frac{1-\cos 12x}{2} + 4 \cdot \frac{1}{2}\sin 12x = a + 5,$

$1 + \cos 12x - \frac{9 - 9\cos 12x}{2} + 2\sin 12x = a + 5,$

$2 + 2\cos 12x - 9 + 9\cos 12x + 4\sin 12x = 2a + 10,$

$11\cos 12x + 4\sin 12x = 2a + 17,$

$\sqrt{137}\left(\frac{11}{\sqrt{137}}\cos 12x + \frac{4}{\sqrt{137}}\sin 12x\right) = 2a + 17,$

$\sin\varphi = \frac{11}{\sqrt{137}}$ deb olsak, natijada quyidagiga ega bo‘lamiz:

$\cos 12x\cos\varphi + \sin 12x\sin\varphi = \frac{2a+17}{\sqrt{137}},\ \cos(12x-1) = \frac{2a+17}{\sqrt{137}}$. Bu yerda $\varphi = arc\ \mathrm{tg}\ \frac{1}{11}$. Oxirgi tenglama $\left|\frac{2a+17}{\sqrt{137}}\right| \leq 1$ shartda yechimga ega. Bundan:

$-1 \leq \frac{2a+17}{\sqrt{137}} \leq 1,\ -\sqrt{137} \leq 2a + 17 \leq \sqrt{137},$

$\frac{-17-\sqrt{137}}{2} \leq a \leq \frac{\sqrt{137}-17}{2}$

a ning bu qiymatlarida $\cos(12x-\varphi) = \frac{2a+17}{\sqrt{137}}$ tenglamani yechamiz.

$12x - \varphi = \pm arc\cos\frac{2a+17}{\sqrt{137}}$,

$12x = arc\,\mathrm{tg}\frac{4}{11} \pm arc\cos\frac{2a+17}{\sqrt{137}} + 2k\pi,\ k \in z;$

$x = \frac{1}{12} arc\,\mathrm{tg}\frac{4}{11} \pm \frac{1}{12} arc\cos\frac{2a+17}{\sqrt{137}} + \frac{k\pi}{6}\ k \in z$

Javob: Agar $a \in \left[\frac{-17-\sqrt{137}}{2}; \frac{\sqrt{137}-17}{2}\right]$ bo'lsa,

$x = \frac{1}{12} arc\,\mathrm{tg}\frac{4}{11} \pm \frac{1}{12} arc\cos\frac{2a+17}{\sqrt{137}} + \frac{k\pi}{6}\ \ k \in z.$

Agar $a \in \left(-\infty; \frac{-17-\sqrt{137}}{2}\right) \cup \left(\frac{-17+\sqrt{137}}{2}; 2\right)$ bo'lsa, $x \in 0$;

Agar $a \in \left[\frac{-17-\sqrt{137}}{2}; \frac{\sqrt{137}-17}{2}\right]$ bo'lsa,

$x = \frac{1}{12} arc\,\mathrm{tg}\frac{4}{11} \pm \frac{1}{12} arc\cos\frac{2a+17}{\sqrt{137}} + \frac{k\pi}{6}\ \ k \in z.$

8. $\sin^4 x - 2\cos^2 x + a^2 = 0$ tenglama yechilsin.

Yechish: $\sin^2 x = 1 - \cos^2 x$ formuladan foydalanamiz.

$(1 - \cos^2 x)^2 - 2\cos^2 x + a^2 = 0$

$1 - 2\cos^2 x + \cos^4 x - 2\cos^2 x + a^2 = 0,$

$\cos^4 x - 4\cos^2 x + a^2 + 1 = 0,$

$\cos^2 x = y$ deb olsak, $y^2 - 4y^2 + a^2 + 1 = 0$ tenglama hosil bo'ladi.

$y_{1,2} = 2 \pm \sqrt{4 - a^2 - 1} = 2 \pm \sqrt{3 - a^2};\ 3 - a^2 \geq 0,$

$a^2 \leq 3,\ -\sqrt{3} \leq a \leq \sqrt{3},\ \cos^2 x = 2 \pm \sqrt{3 - a^2},$

$\frac{1+\cos 2x}{2} = 2 \pm \sqrt{3 - a^2},$

$1 + \cos^2 x = 4 + 2\sqrt{3 - a^2},\ \cos^2 x = 3 + 2\sqrt{3 - a^2}$

$2x = \pm arc\cos(3 - 2\sqrt{3 - a^2}) + 2k\pi,\ k \in z;$

$x = \pm\frac{1}{2} arc\cos(3 - 2\sqrt{3 - a^2}) + k\pi;\ k \in z.$

Javob: Agar $a \in [-\sqrt{3}; \sqrt{3}]$ bo'lsa,

$x = \pm arc\cos(3 - 2\sqrt{3 - a^2}) + k\pi,\ k \in z;$ agar $a \in [-\sqrt{3}; \sqrt{3}]$ bo'lsa, $\emptyset$.

9. $\sin^3 x + \cos^3 x + a\sin\left(x + \frac{\pi}{4}\right) = 0$ tenglama yechilsin.

Yechish: $\sin^3 x + \cos^3 x = (\sin x + \cos x)(\sin^2 - \sin x\cos x + \cos^2 x) = (\sin x + \cos x)(1 - \sin x\cos x)$ va $\sin\left(x + \frac{\pi}{4}\right) = \sin x \cdot \cos\frac{\pi}{4} + \cos x \cdot \sin\frac{\pi}{4} = \cos x \cdot \frac{\sqrt{2}}{2} + \sin x \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(\sin x + \cos x)$ lardan foydalanamiz.

Demak, $(\sin x + \cos x)(1 - \sin x\cos x) + \frac{\sqrt{2}}{2}a(\sin x + \cos x) = 0,\ (\sin x + \cos x)\left(1 - \sin x\cos x + \frac{\sqrt{2}}{2}a\right) = 0.$

Bundan $\sin x + \cos x = 0$ va $1 - \sin x\cos x + \frac{\sqrt{2}}{2}a = 0$ tenglamalar kelib chiqadi. Ularni yechamiz:

1) $\sin x + \cos x = 0, \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right) = 0,$

$\sin x\cos\frac{\pi}{4} + + \cos x\sin\frac{\pi}{4} = 0,$

$\sin\left(x + \frac{\pi}{4}\right) = 0,\ x + \frac{\pi}{4} = k\pi,\ x = -\frac{\pi}{4} + k\pi,\ k \in z;$

2) $1 - \sin x \cos x + \frac{\sqrt{2}}{2} a = 0,\ \sin x \cos x = 1 + \frac{\sqrt{2}}{2} a,$

$\frac{1}{2} \sin 2x = \frac{2 + \sqrt{2}a}{2},\ \sin 2x = 2 + \sqrt{2}a.$

Bu tenglama $-1 \le 2 + \sqrt{2}a \le 1$ shartda yechimga ega. Bu tengsizlikni yechamiz.

$-3 \le \sqrt{2}a \le -1,\ -\frac{3}{\sqrt{2}} \le a \le -\frac{1}{\sqrt{2}}$. a ning bu tengsizlikni qanoatlantiradigan qiymatlarida $\sin 2x = 2 + \sqrt{2}a$ tenglamani yechamiz.

$\sin 2x = 2 + \sqrt{2}a,\ 2x = (-1)^k\ arc\ \sin(2 + \sqrt{2}a) + k\pi,$

$x = \frac{(-1)^k}{2}\ arc\ \sin(2 + \sqrt{2}a) + \frac{k\pi}{2}\ k \in z$

Javob: $a \in \left[-\infty; -\frac{3}{\sqrt{2}}\right] \cup \left[-\frac{1}{\sqrt{2}}; \infty\right]$ bo‘lsa, $\left\{-\frac{k\pi}{4} + k\pi,\ k \in z\right\};$

$a \in \left[-\frac{3}{\sqrt{2}}; -\frac{1}{\sqrt{2}}\right]$ bo‘lsa, $\left\{-\frac{\pi}{4} + k\pi;\ \frac{(-1)^k}{2}\ arc\ \sin\ (2 + \sqrt{2}a) + \frac{k\pi}{2},\ k \in z\right\}.$

10. $\sin x + \sin \frac{3x}{2} = a \sin \frac{x}{2}$ tenglama yechilsin.

Yechish: $\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha$ formuladan foydalanib, $\sin \frac{3x}{2}$ ifodani yoyamiz. Demak, $\sin 3\frac{3x}{2} = 3 \sin \frac{x}{2} - 4 \sin^3 \frac{3x}{2}$. Buni o‘rniga qo‘yamiz. $\sin x + 3 \sin \frac{x}{2} - 4\sin^3 \frac{x}{2} = a \sin \frac{x}{2}$ $\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$ bo‘lganligi uchun oxirgi tenglama quyidagi ko‘rinishga keladi:

$2 \sin \frac{x}{2} \cos \frac{x}{2} + 3 \sin \frac{x}{2} - 4\sin^3 \frac{x}{2} - a \sin \frac{x}{2} = 0$

$\sin \frac{x}{2} \left(2 \cos \frac{x}{2} + 3 - 4\sin^2 \frac{x}{2} - a\right) = 0$ yoki

$$\left[\begin{array}{l} \sin \frac{x}{2} = 0, \\ 2 \cos \frac{x}{2} - 4\left(1 - \cos^2 \frac{x}{2}\right) - a + 3 = 0 \end{array}\right.$$

$$\left[\begin{array}{l} \sin \frac{x}{2} = 0, \\ 4 \cos^2 \frac{x}{2} + 2 \cos \frac{x}{2} - 1 - a = 0 \end{array}\right.$$

Majmuaning birinchi tenglamasidan $\frac{x}{2} = k\pi$ yoki $x = 2k\pi,\ k \in z,\ a \in R$. Majmuaning ikkinchi tenglamasida $\cos \frac{x}{2} = t$ deb olsak, natijada $4t^2 + 2t - 1 - a = 0$ tenglamani hosil qilamiz. Bu yerda $t \in [-1; 1]$ bo‘lishi kerak.

$t_{1,2} = \frac{t \pm \sqrt{1 + 4a + 4}}{4} = \frac{-1 \pm \sqrt{4a + 5}}{4};\ t_1 = \frac{-1 - \sqrt{4a + 5}}{4},\ t_2 = \frac{-1 + \sqrt{4a + 5}}{4};$

Endi $-1 \le t_1 \le 0$ va $-1 \le t_2 \le 1$ tengsizliklarni yechamiz.

1) $-1 \le \frac{-1 - \sqrt{4a + 5}}{4} < 0,\ -4 \le -1 - \sqrt{4a + 5} < 0,$

$-3 \le -\sqrt{4a + 5} < 0,$

$0 < \sqrt{4a + 5} \le 3,\ 0 < 4a + 5 \le 9,$

$-5 < 4a \le 4,\ -\frac{5}{4} < a \le 1.$

Agar $4a + 5 \ge 0$, ya’ni $a \ge -\frac{5}{4}$ ni e’tiborga olsak, $a \in (-1; 1]$.

2) $-1 \le \frac{-1 + \sqrt{4a + 5}}{4} \le 1,$

$-4 \le -1+\sqrt{4a+5} \le 4-3 \le \sqrt{4a+5} \le 5,$

$9 \le 4a+5 \le 25,\ 4 \le 4a \le 20,\ 1 \le a \le 5,\ a \in [1;5].$

$\cos\frac{x}{2} = t_1,\ \cos\frac{x}{2} = \frac{-1-\sqrt{4a+5}}{4},\ x = 4k\pi \pm 2arc\cos\frac{-1-\sqrt{4a+5}}{4},\ k \in z$

agar $a \in \left[-\frac{5}{4};\ 1\right]$ bo'lsa, $\cos\frac{x}{2} = t_2,\ \cos\frac{x}{2} = \frac{\sqrt{4a+5}-1}{4},$

Javob: $x = 4k\pi \pm 2arc\cos\frac{\sqrt{4a+5}-1}{4},\ k \in z,\ a \in \left[-\frac{5}{4};\ 5\right].$

11. $\sin 2x + 3\cos 2x = a$ tenglama yechilsin.

Yechish: Tenglamaning chap tomonini $\sqrt{10}$ ga ko'paytiramiz va bo'lamiz.

$\sin 2x + 3\cos 2x = a,\ \sqrt{10}\left(\frac{1}{\sqrt{10}}\sin 2x + \frac{3}{\sqrt{10}}\cos 2x\right) = a,\ \frac{1}{\sqrt{10}} = \sin\varphi$ deb olsak, $\frac{3}{\sqrt{10}} = \cos\varphi$ bo'lishi ravshan. Demak, tenglamaning chap tomoni quyidagi ko'rinishga keladi:

$\sqrt{10}\ (\sin 2x\sin\varphi + \cos 2x\cos\varphi) = a,\ \sqrt{10}\cos(2x-\varphi) = a.$

Bu yerda $\varphi = arc\ \mathrm{tg}\frac{1}{3}$ oxirgi tenglamadan $\cos(2x-\varphi) = \frac{a}{\sqrt{10}}$ ni yozamiz. Bu tenglama $-1 \le \frac{a}{\sqrt{10}} \le 1$, ya'ni $-\sqrt{10} \le a \le \sqrt{a}$ shartda yechimga ega. Shunday qilib $2x - arc\ \mathrm{tg}\frac{1}{3} = \pm arc\cos\frac{a}{\sqrt{10}} + k\pi,\ k \in z$. Bu yerda $|a| \le \sqrt{10}$. Endi yuqoridagi tenglikdan x ni topamiz:

$2x = arc\ \mathrm{tg}\frac{1}{3} \pm arc\cos\frac{a}{\sqrt{10}} + 2k\pi,\ x = \frac{1}{2}arc\ \mathrm{tg}\frac{1}{3} \pm \frac{1}{2}arc\cos\frac{a}{\sqrt{10}} + k\pi,\ k \in z.$

Javob: $x = \frac{1}{2}arc\ \mathrm{tg}\frac{1}{3} \pm \frac{1}{2}arc\cos\frac{a}{\sqrt{10}} + k\pi,\ k \in z,$ agar $-\sqrt{10} \le a \le \sqrt{10}$ bo'lsa; $\emptyset$ agar $a \in (-\infty;\ -\sqrt{10}) \cup (\sqrt{10};\ +\infty)$

12. $2\cos^2 6x - 9\sin^2 6x + 4\sin 6x\cos 6x = a + 5$ tenglama yechilsin.

Yechish: $\cos^2 6x = \frac{1+\cos 12x}{2},\ \sin^2 6x = \frac{1-\cos 12x}{2},\ \sin 6x\cos 6x = \frac{1}{2}\sin 12x$ lardan foydalanamiz va quyidagi tenglamaga ega bo'lamiz:

$2\cdot\frac{1+\cos 12x}{2} - 9\cdot\frac{1-\cos 12x}{2} + 4\cdot\frac{1}{2}\sin 12x = a + 5,$

$2(1+\cos 12x) - 9(1-\cos 12x) + 4\sin 12x = 2a + 10,$

$2 + 2\cos 12x - 9 + 9\cos 12x + 4\sin 12x = 2a + 10,$

$11\cos 12x + 4\sin 12x = 2a + 17.$ Bu tenglamaning chap tomonini $\sqrt{137}$ ga ko'paytiramiz va bo'lamiz.

$\sqrt{137}\left(\frac{4}{\sqrt{137}}\cos 12x + \frac{4}{\sqrt{137}}\sin 12x\right) = 2a + 17,$

$\sqrt{137}\ (\cos 12x\cos\varphi + \sin 12x\sin\varphi) = 2a + 17,$

$\sqrt{137}\cos(12x-\varphi) = 2a + 17.$

Bu yerda $\varphi = arc\ \mathrm{tg}\frac{4}{11}$. Oxirgi tenglamadan $\cos(12x-\varphi)\ \frac{2a+17}{\sqrt{137}}$ ni yozamiz. Bu tenglama $-1 \le \frac{2a+17}{\sqrt{137}} \le 1$ shartda yechimga ega. Bu tengsizlikni yechamiz:

$$\begin{cases} \frac{2a+17}{\sqrt{137}} \ge -1 \\ \frac{2a+17}{\sqrt{137}} \le 1 \end{cases};\quad \begin{cases} \frac{2a+17}{\sqrt{137}} + 1 \ge 0 \\ \frac{2a+17}{\sqrt{137}} - 1 \le 0 \end{cases};$$

$\begin{cases} 2a + 17 + \sqrt{137} \geq 0 \\ 2a + 17 - \sqrt{137} \leq 0 \end{cases}$; $\begin{cases} a \geq -\dfrac{17 - \sqrt{137}}{2} \\ a \leq \dfrac{\sqrt{137} - 17}{2} \end{cases}$;

$\dfrac{-\sqrt{137} - 17}{2} \leq a \ \dfrac{\sqrt{137} - 17}{2}$

a ning bu tengsizlikni qanoatlantiradigan qiymatlarida quyidagilarga ega bo‘lamiz:

$12x - \varphi = \pm arc\cos\dfrac{2a + 17}{\sqrt{137}} + k\pi,\ k \in z$

$12x = arc\,\mathrm{tg}\dfrac{4}{11} \pm arc\cos\dfrac{2a + 17}{\sqrt{137}} + 2k\pi,\ k \in z$

Bundan x ni topamiz.

$x = \dfrac{1}{12} arc\,\mathrm{tg}\dfrac{4}{11} \pm \dfrac{1}{12} arc\cos\dfrac{2a + 17}{\sqrt{137}} + \dfrac{k\pi}{6},\ k \in z.$

Javob: Agar $a \in \left[\dfrac{-\sqrt{137} - 17}{2};\ \dfrac{\sqrt{137} - 17}{2}\right]$ bo‘lsa,

$x = \dfrac{1}{12} arc\,\mathrm{tg}\dfrac{4}{11} \pm \dfrac{1}{12} arc\cos\dfrac{2a + 17}{\sqrt{137}} + \dfrac{k\pi}{6},\ k \in z.$

Agar a $\in \left(-\infty;\ \dfrac{-\sqrt{137} - 17}{2}\right) \cup \left(\dfrac{-\sqrt{137} - 17}{2};\ +\infty\right)$ bo‘lsa, $\emptyset$.

13. $\sin^4 x - 2\cos^2 x + a^2 = 0$ yechilsin.

Yechish: $\sin^4 x - 2\cos^2 x = -a^2$, $(\sin^2 x)^2 - 2\cos^2 x = -a^2$,

$\left(\dfrac{1 - \cos 2x}{2}\right)^2 - (1 + \cos 2x) = -a^2$, $\dfrac{(1 - \cos 2x)}{4} - 1 - \cos 2x = -a^2$

$1 - 2\cos 2x + \cos^2 2x - 4 - 4\cos 2x = -4a^2$,

$\cos^2 2x - 6\cos 2x = 3 - 4a^2$, $\cos^2 2x - 6\cos 2x - (3 - 4a^2) = 0$,

$(\cos 2x)_{1,2} = 3 \pm \sqrt{9 + 3 - 4a^2} = 3 \pm \sqrt{12 - 4a^2} = 3 \pm 2\sqrt{3 - a^2}$.

Demak, biz $\cos 2x = 3 - 2\sqrt{3 - a^2}$ va $\cos 2x = 3 + 2\sqrt{3 - a^2}$ lardan iborat ikkita tenglamaga ega bo‘ldik. Quyida bu tenglamalarni yechamiz.

1) $\cos 2x = 3 - 2\sqrt{3 - a^2}$ tenglama $-1 \leq 3 - 2\sqrt{3 - a^2} \leq 1$

shartda yechimga ega. Bu qo‘sh tengsizlikni yechamiz.

$-4 \leq -2\sqrt{3 - a^2} \leq -2,\ 2 \leq 2\sqrt{3 - a^2} \leq 4,1 \leq \sqrt{3 - a^2} \leq 2.$

Bu quyidagi sistemaga teng kuchli:

$\begin{cases} \sqrt{3 - a^2} \geq \\ \sqrt{3 - a^2} \leq \end{cases}$; $\begin{cases} 3 - a^2 \geq 1 \\ 3 - a^2 \leq 4 \end{cases}$; $\begin{cases} a^2 \leq 2 \\ a^2 \geq -1 \end{cases}$; $\begin{cases} -\sqrt{2} \leq a \leq 2 \\ a \in R \end{cases}$; $(-\sqrt{2};\ \sqrt{2})$.

$\cos 2x = 3 - 2\sqrt{3 - a^2},\ 2x = \pm arc\cos(3 - 2\sqrt{3 - a^2}) + 2k\pi,$

$x = \pm\dfrac{1}{2} arc\cos(3 - 2\sqrt{3 - a^2}) + k\pi.$

Javob: $a \in [-\sqrt{2};\ \sqrt{2}]$ bo‘lsa,

$x = \pm\dfrac{1}{2} arc\cos(3 - 2\sqrt{3 - a^2}) + k\pi;$

Agar, $a \in [-\sqrt{2};\ \sqrt{2}]$ bo‘lsa, y xolda $\emptyset$.

Yuqorida ko‘rib o‘tilgan misollardan parametr qatnashgan trigonometrik tenglamalar parametrning ba‘zi bir qiymatlarida yechimga ega, ba‘zi bir qiymatlarida esa yechimga ega emasligini ko‘rdik. Parametrning berilgan tenglamani yechimga ega yoki yechimga ega qilmaydigan qiymatlarini ajrata olish o‘quvchidan mushohada

qilish va ijodiy izlanishni talab qiladi. Buning natijasida o'quvchilarning ijodiy fikrlash qobiliyatlari yanada rivojlanadi va matematik ta'lim jarayoni samarali bo'ladi.

Adabiyotlar

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STUDYING TRIGONOMETRIC EQUATIONS WITH PARAMETERS IN GENERAL SECONDARY EDUCATION SCHOOLS

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It is known that in recent years, great importance has been placed on teaching mathematics at all levels of the education system, including general secondary schools. In particular, thoroughly conveying topics rich in practical applications of mathematics has become one of the primary tasks. Solving many practical problems leads to the formulation and solution of equations that are part of the mathematical model, including trigonometric equations. Among trigonometric equations, those that involve parameters hold special significance. Therefore, this article presents trigonometric equations with parameters and methods for solving them, supported by sufficient examples.

When solving trigonometric equations involving parameters, some transformations are performed using trigonometric formulas and basic trigonometric properties. If such substitutions are based on theorems about the equal power of equations, then the solution of the resulting equation is the solution of the given equation. Otherwise, the solutions should be checked. The concept of possible values of a parameter is important when solving trigonometric equations involving a parameter

Any trigonometric equation can be transformed into the simplest forms such as $\sin[f(x)] = a\ (|a| \le 1)$, $\cos[f(x)] = a\ (|a| \le 1)$, $\text{tg}[f(a)] = a$, and $ctg[f(x)] = a$ after certain substitutions. Thus, the article includes their general solutions.

Additionally, it is mentioned that the concept of possible values of the parameter is crucial when solving trigonometric equations with parameters. As an example, for the equations $\sin\left(2x + \frac{\pi}{3}\right) = a$, $\cos\left(5x + \frac{\pi}{3}\right) = a$ the important values of the parameter a must satisfy the inequality $|a| \le 1$. In contrast, for the equations $tg\left(\frac{1}{3}x + \frac{\pi}{6}\right) = a$ va $ctg\left(2x + \frac{\pi}{4}\right) = a$, the parameter has been emphasized that for some trigonometric equations involving a parameter, determining the possible values of the parameter before solving the equation is necessary, while for others, the parameter values are determined at the end of the solution process can take any value.

The article provides sufficient examples with solutions for each type of trigonometric equation with parameters. It concludes that solving these and similar equations enhances students' creative thinking abilities, which, in turn, contributes to more effective mathematical education.

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